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THREE-DIMENSIONAL CURVED CRACK IN AN ELASTIC BODY

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Abstract—The boundary integral equations, which give the relation between the crack opening displacement and traction on the surface of a crack embedded in an infinite isotropic elastic body are formulated. The integral equations are transformed into spherical and cylindrical coordinates in the cases of cracks curved in the shape of spherical and cylindrical surfaces respectively, so that these boundary integral equations may be converted into a system of algebraic equations by the boundary element method. The dependence of stress-intensity factors on the curvature of crack has been numerically calculated for the spherical crack with circular contour under a constant load.

1. INTRODUCTION

The boundary integral equations for the investigation of the state of stress in an infinite body with a crack are formulated. Special attention is devoted to cracks curved in the shape of a spherical and cylindrical surface. Cracks of this kind can appear in bodies as a result of propagation of a flat three-dimensional crack subjected to mixed mode I, II and III loads when the trajectory of crack growth is curved, i.e. it does not lie in the plane of initial crack[1]. Up to now, works on curved cracks have been restricted to two-dimensional cracks[2, 3]. Cotterell and Rice[3] have investigated the dependence of stress-intensity factors K_I and K_{II} on the curvature of cracks in two-dimensional case.

In this paper the dependence of the stress-intensity factors on the curvature of three-dimensional crack is discussed using the boundary element method (BEM)[4]. The boundary integral equations (BIE) are formulated by Somigliana method. These integral equations determine the dependence between the crack opening displacement and tractions on the surface of a crack embedded in an infinite elastic medium[7]. In the case of cracks curved in the shape of a spherical and cylindrical surfaces, the BIE are rewritten in terms of spherical and cylindrical coordinates respectively. Having known the crack opening displacement, as a solution of the derived BIE, one can use the Somigliana formula for evaluation of the displacement field at an arbitrary inner point of the body. Numerical calculation of the dependence of stress-intensity factors on the curvature of spherical crack with circular contour has been carried out in the case of constant loading on the crack surface.

The conventional 3-dimensional boundary integral approach[4] cannot be used in this problem because of non-unique dependence of the solution on the state of loading of the crack surface[11]. The general formulation like that of Lachat and Watson[12] in which quadratic isoparametric elements are used to approximate curved surface can be used in the case of a finite body.

2. FORMULATION OF BOUNDARY INTEGRAL EQUATION

In this section the boundary integral equations for unknown displacements on the surface of a crack in infinite elastic medium are concisely derived.

Let us consider the infinite elastic medium. The classical equations of elastostatics for displacement field $u_i(\mathbf{x})$ are Navier's equations

$$u_{i,ij} + (1-2\nu)u_{j,ii} + \frac{1-2\nu}{\mu}X_j = 0, \quad (i, j = 1, 2, 3) \tag{1}$$

where $X_i(\mathbf{x})$ are components of body forces, ν is Poisson's ratio and μ is Lame's constant. The usual notation is used. Partial differentiation with respect to Descartes coordinates is indicated by a comma in front of the suffix and repeated suffixes are to be summed.

The response to the concentrated body force $X_i(\mathbf{x}) = e_i\delta(\mathbf{x}-\boldsymbol{\eta})$ acting at point $\boldsymbol{\eta}$ is the displacement field

$$u_j(\mathbf{x}) = U_{ij}(\boldsymbol{\eta}-\mathbf{x})e_i \tag{2}$$

where

$$U_{ij}(\boldsymbol{\eta}-\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)r}[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}] \tag{3}$$

is the familiar Kelvin's solution and

$$r^2 = (x_i-\eta_i)(x_i-\eta_i).$$

Tractions $t_i(\mathbf{x})$ on an arbitrary surface S with the unit normal $\mathbf{n}(\mathbf{x})$ may be expressed in terms of the displacement field $u_i(\mathbf{x})$ by Hooke's law

$$t_i(\mathbf{x}) = \sigma_{ij}(\mathbf{x})n_j(\mathbf{x}) = \hat{T}_{ik}(n_x, \partial_x)u_k(\mathbf{x}) \tag{4}$$

where

$$\hat{T}_{ik}(n_x, \partial_x) \equiv \mu\left[n_l(\mathbf{x})\,\delta_{ik}\partial_l + \frac{2\nu}{1-2\nu}n_i(\mathbf{x})\partial_k + n_k(\mathbf{x})\partial_i\right],$$

$$\partial_i \equiv \frac{\partial}{\partial x_i},$$

is the so-called "stress operator."

Inserting (2) into (4), we obtain tractions

$$t_i(\mathbf{x}) = \hat{T}_{ik}(n_x, \partial_x)U_{jk}(\boldsymbol{\eta}-\mathbf{x})e_j = T_{ji}(n_x, \boldsymbol{\eta}-\mathbf{x})e_j \tag{5}$$

corresponding to the body force $X_i(\mathbf{x}) = e_i\delta(\mathbf{x}-\boldsymbol{\eta})$. The fundamental tractions $T_{ij}(n_x, \boldsymbol{\eta}-\mathbf{x})$ associated with the fundamental displacement field $U_{jk}(\boldsymbol{\eta}-\mathbf{x})$ are given by $T_{ji}(n_x, \boldsymbol{\eta}-\mathbf{x}) \equiv$

$$\hat{T}_{ik}(n_x, \partial_x)U_{jk}(\boldsymbol{\eta}-\mathbf{x}) = -\frac{1-2\nu}{8\pi(1-\nu)r^2}\left\{\frac{\partial r}{\partial n}\left[\delta_{ij} + 3\frac{r_{,i}r_{,j}}{1-2\nu}\right] + n_j(\mathbf{x})r_{,i} - n_i(\mathbf{x})r_{,j}\right\}$$

$$\frac{\partial r}{\partial n} = r_{,i}n_i(\mathbf{x}). \tag{6}$$

The displacement vector $u_i(\mathbf{x})$ at an arbitrary point $\mathbf{x}$ of the region D can be expressed in terms of surface tractions $t_k(\boldsymbol{\eta})$ and displacements $u_k(\boldsymbol{\eta})$ by the Somigliana formula

$$u_i(\mathbf{x}) = \int_S [t_k(\boldsymbol{\eta})U_{ik}(\boldsymbol{\eta}-\mathbf{x}) - u_k(\boldsymbol{\eta})T_{ik}(n_\eta, \boldsymbol{\eta}-\mathbf{x})]\,dS_\eta \tag{7}$$

where S is the boundary surface of the region D.

In considered problem, the surface S consists of the crack surface $S = S_{cr}^+ \cup S_{cr}^-$, where $S_{cr}^\pm$ are parts of Ljapunov surfaces differing only in their unit normals

$$n^+(\boldsymbol{\eta}) = -n^-(\boldsymbol{\eta}). \tag{8}$$

The relation of symmetry

$$U_{ik}(\boldsymbol{\eta}, \mathbf{x})|_{S_{cr}^+} = U_{ik}(\boldsymbol{\eta}, \mathbf{x})\Big|_{S_{cr}^-}$$

follows directly from the equation (3). We shall consider the symmetric crack loading

$$t_k(\boldsymbol{\eta})|_{S^+_{cr}} = -\, t_k(\boldsymbol{\eta})|_{S^-_{cr}}$$

Then the first integral in eqn (7) vanishes, i.e.

$$\int_S t_k(\boldsymbol{\eta})\, U_{ik}(\boldsymbol{\eta}-\mathbf{x})\,\mathrm{d}S_\eta = 0. \tag{9}$$

The displacement field $u_i(\mathbf{x})$ at any point $\mathbf{x} \in D$ can be expressed through displacements on the crack surface or through crack opening displacement $\Delta u_k(\boldsymbol{\eta})$

$$\begin{aligned} u_i(\mathbf{x}) &= -\int_S u_k(\boldsymbol{\eta})\, T_{ik}(n_\eta, \boldsymbol{\eta}-\mathbf{x})\,\mathrm{d}S_\eta \\ &= -\int_{S^+_{cr}} \Delta u_k(\boldsymbol{\eta})\, T_{ik}(n_\eta, \boldsymbol{\eta}-\mathbf{x})\,\mathrm{d}S_\eta \end{aligned} \tag{10}$$

where

$$\Delta u_k(\boldsymbol{\eta}) \equiv u_k(\boldsymbol{\eta})|_{S^+_{cr}} - u_k(\boldsymbol{\eta})|_{S^-_{cr}}.$$

The relation of symmetry

$$T_{ik}(n_\eta, \boldsymbol{\eta}-\mathbf{x})|_{S^+_{cr}} = -\, T_{ik}(n_\eta, \boldsymbol{\eta}-\mathbf{x})|_{S^-_{cr}}$$

that follows from eqns (6) and (8), has been used in eqn (10).

Let $x \to \zeta \in S$ in eqn (10). According to limit behaviour of the double layer potentials [6] we obtain the boundary integral equation

$$\frac{1}{2}\, u_i(\zeta) = -\int_S u_k(\boldsymbol{\eta})\, T_{ik}(n_\eta, \boldsymbol{\eta}-\zeta)\,\mathrm{d}S_\eta. \tag{11}$$

This equation, however, cannot be used to calculate the displacements $u_i(\boldsymbol{\eta})$ on the surface of the crack, as this one does not depend on the loading of the crack surface.

Applying the stress operator $T_{ik}(n_x, \partial_x)$ to eqn (10) and using the relation (6), we obtain the following expression for tractions at $\mathbf{x} \in D$

$$t_l(\mathbf{x}) = -\, \hat{T}_{li}(n_x, \partial_x) \int_S u_k(\boldsymbol{\eta})\, \hat{T}_{kj}(h_\eta, \partial_\eta)\, U_{ij}(\boldsymbol{\eta}-\mathbf{x})\,\mathrm{d}S_\eta. \tag{12}$$

In order to calculate the unknown surface displacements, the limit passage $\mathbf{x} \to \zeta \in S$ must be performed in eqn (12). Some steps of this operation are included in the Appendix. The result is the following boundary integral equation

$$\begin{aligned} t_l(\zeta) = \frac{\mu n_p(\zeta)}{8\pi(1-\nu)} \int_{S^+_{cr}} \frac{1}{r^2} \{ & 4\nu\delta_{lp}\Delta\kappa^i_{ik} r_{,k} + 3 r_{,i} r_{,k} (\Delta\kappa^i_{pk} r_{,l} + \Delta\kappa^i_{lk} r_{,p}) \\ & + (1-2\nu)[\Delta\kappa^l_{ip} r_{,l} + \Delta\kappa^i_{il} r_{,p} + r_{,k}(\Delta\kappa^l_{pk} + \Delta\kappa^p_{lk})]\}\,\mathrm{d}S_\eta \end{aligned} \tag{13}$$

where

$$\Delta\kappa^i_{jk} \equiv n_j(\boldsymbol{\eta})\Delta u_{i,k}(\boldsymbol{\eta}) - n_k(\boldsymbol{\eta})\Delta u_{i,j}(\boldsymbol{\eta})$$

$$r = |\boldsymbol{\zeta} - \boldsymbol{\eta}|.$$

The last integral equation determines the dependence between tractions $t_l(\zeta)$ on the crack surface and crack opening displacement $\Delta u_i(\boldsymbol{\eta})$. The integrals in (13) exist in the sense of the Cauchy principle value[6].

The integral equations have been derived for an infinite isotropic elastic medium containing a symmetrically loaded crack with arbitrary Ljapunov surfaces $S_{cr}^{\pm}$. These equations have the simplest form when the cracks lie in a plane, e.g. (x_1, x_2). Interaction between two penny-shaped cracks was investigated[7] by using these equations. The rest of this paper is devoted to axially-symmetric cracks.

3. SPHERICAL CRACK WITH CIRCULAR CONTOUR

Spherical coordinates $(\rho, \vartheta, \varphi)$ are appropriate in this particular case. The Descartes coordinates (η_1, η_2, η_3) of arbitrary point $\boldsymbol{\eta} \in S_{cr}^{+}$ are given by the transformation relations

$$\eta_1 = \rho \sin\vartheta \cos\varphi$$

$$\eta_2 = \rho \sin\vartheta \sin\varphi$$

$$\eta_3 = \rho \cos\vartheta$$

and $\rho = \text{const}$,

$$\varphi \in \langle 0; 2\pi \rangle, \vartheta \in \langle 0; \vartheta_0 \rangle.$$

Let the spherical coordinates of the point $\boldsymbol{\zeta} \in S$ be (ρ, θ, ϕ). Then, we can write the following apparent relations

$$\frac{r}{\rho} = [(\sin\theta\cos\phi - \sin\vartheta\cos\varphi)^2 + (\sin\theta\sin\phi - \sin\vartheta\sin\varphi)^2 + (\cos\theta - \cos\vartheta)^2]^{1/2}$$

$$r_{,i} = \frac{\zeta_i - \eta_i}{r} = \frac{\rho}{r}(\sin\theta\cos\phi - \sin\vartheta\cos\varphi, \sin\theta\sin\phi - \sin\vartheta\sin\varphi, \cos\theta - \cos\vartheta) \qquad (15)$$

$$\mathbf{n} = (\vartheta, \varphi) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta).$$

The Descartes components of the gradient of the crack opening displacement $\Delta u_{i,k}(\boldsymbol{\eta})$, defined on the crack surface, can be expressed in terms of $\partial\Delta u_i/\partial\vartheta$ and $\partial\Delta u_i/\partial\varphi$ as

$$\begin{pmatrix} \Delta u_{i,1} \\ \Delta u_{i,2} \\ \Delta u_{i,3} \end{pmatrix} = \begin{pmatrix} A_{11} & \frac{1}{\rho}\cos\vartheta\cos\varphi & -\frac{\sin\varphi}{\rho\sin\vartheta} \\ A_{21} & \frac{1}{\rho}\cos\vartheta\sin\varphi & \frac{\cos\varphi}{\rho\sin\vartheta} \\ A_{31} & -\frac{1}{\rho}\sin\vartheta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\partial\Delta u_i}{\partial\vartheta} \\ \frac{\partial\Delta u_i}{\partial\varphi} \end{pmatrix} \qquad (16)$$

The integration element $\mathrm{d}S_\eta = \rho^2 \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi$. In such a way all the quantities of equation (13) are expressed as functions dependent on two variables ϑ, φ. An appropriate choice of nodal points is important for numerical solution of these integral equations by BEM. The grid of nodal points is not regular in the band $\vartheta \in \langle 0; \vartheta_0 \rangle$, $\varphi \in \langle 0; 2\pi \rangle$. That is why the new variables ξ_1, ξ_2 are introduced. The coordinate of any inner point of the surface element are

$$\vartheta = \sum_{\alpha=1}^{4} N_\alpha(\xi_1, \xi_2)\vartheta_\alpha \qquad (17a)$$

$$\varphi = \sum_{\alpha=1}^{4} N_\alpha(\xi_1, \xi_2)\varphi_\alpha \qquad (17b)$$

$$\xi_1, \xi_2 \in \langle -1; 1 \rangle$$

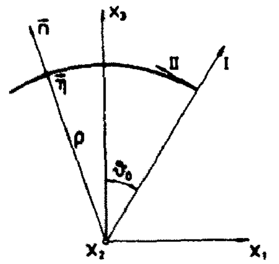

Fig. 1. Spherical crack with circular contour.

where ϑ_α, φ_α are coordinates of the αth nodal point of the surface element and

$$N_1=\frac{1}{4}(1+\xi_1)(1+\xi_2),\quad N_3=\frac{1}{4}(1-\xi_1)(1-\xi_2)$$

$$N_2=\frac{1}{4}(1-\xi_1)(1+\xi_2),\quad N_4=\frac{1}{4}(1+\xi_1)(1-\xi_2) \tag{18}$$

are so-called shape function. Transformation matrix J^{-1}, determining the transformation

$$\begin{pmatrix} \dfrac{\partial \Delta u_i}{\partial \vartheta} \\ \dfrac{\partial \Delta u_i}{\partial \varphi} \end{pmatrix} = J^{-1} \begin{pmatrix} \dfrac{\partial \Delta u_i}{\partial \xi_1} \\ \dfrac{\partial \Delta u_i}{\partial \xi_2} \end{pmatrix} \tag{19}$$

is given by

$$J^{-1}=\frac{1}{|J|}\begin{pmatrix} \dfrac{\partial \varphi}{\partial \xi_2} & -\dfrac{\partial \varphi}{\partial \xi_1} \\ -\dfrac{\partial \vartheta}{\partial \xi_2} & \dfrac{\partial \vartheta}{\partial \xi_1} \end{pmatrix} \tag{20}$$

where

$$|J| = \det J.$$

From eqns (20) and (17)

$$J^{-1}=\frac{1}{|J|}\begin{pmatrix} \varphi_\alpha \dfrac{\partial N_\alpha}{\partial \xi_2} & -\varphi_\alpha \dfrac{\partial N_\alpha}{\partial \xi_1} \\ -\vartheta_\alpha \dfrac{\partial N_\alpha}{\partial \xi_2} & \vartheta_\alpha \dfrac{\partial N_\alpha}{\partial \xi_1} \end{pmatrix}. \tag{21}$$

A repeated greek suffix is understood to be summed over the values 1, 2, 3, 4.

The unknown crack opening $\Delta u_i(\xi_1, \xi_2)$ are approximated over the surface element by the polynomials $N_\alpha(\xi_1, \xi_2)$ as

$$\Delta u_i(\xi_1, \xi_2)=\sum_{\alpha=1}^{4} N_\alpha(\xi_1, \xi_2)\Delta u_i^\alpha \tag{22}$$

where Δu_i^α are unknown values of the crack opening at the αth nodal point of the surface element. By using this approximation the system of the integral equations is converted into a system of a algebraic equations for the crack opening at all the nodal points on the crack surface.

From eqns (19), (21) and (22)

$$\begin{pmatrix} 0 \\ \dfrac{\partial \Delta u_i}{\partial \vartheta} \\ \dfrac{\partial \Delta u_i}{\partial \varphi} \end{pmatrix} = \frac{D_{\alpha\beta}\Delta u_i^{\beta}}{|J|}\begin{pmatrix} 0 \\ \varphi_\alpha \\ -\vartheta_\alpha \end{pmatrix} \tag{23}$$

where

$$D_{\alpha\beta} \equiv \frac{\partial N_\alpha}{\partial \xi_2}\frac{\partial N_\beta}{\partial \xi_1} - \frac{\partial N_\alpha}{\partial \xi_1}\frac{\partial N_\beta}{\partial \xi_2}. \tag{24}$$

Finally, from (16) and (23)

$$\Delta u_{i,k} = \frac{1}{|J|} B_k{}^{\beta}\Delta u_i{}^{\beta} \tag{25}$$

where the vector $B_k{}^{\beta}$ takes the form

$$\begin{pmatrix} \beta_1{}^{\beta} \\ B_2{}^{\beta} \\ B_3{}^{\beta} \end{pmatrix} = \frac{D_{\alpha\beta}}{\rho}\begin{pmatrix} \varphi_\alpha \cos\vartheta\cos\varphi + \vartheta_\alpha \dfrac{\sin\varphi}{\sin\vartheta} \\ \varphi_\alpha \cos\vartheta \sin\vartheta - \vartheta_\alpha \dfrac{\cos\varphi}{\sin\vartheta} \\ -\varphi_\alpha \sin\vartheta. \end{pmatrix}. \tag{26}$$

The integration element $dS_\eta = \rho^2 \sin\vartheta\, d\vartheta\, d\varphi$ is changed into $\rho^2 \sin\vartheta |J|\, d\xi_1\, d\xi_2$ by $\vartheta, \varphi \to \xi_1, \xi_2$. At last, the integral equations (13) may be rewritten as

$$\begin{aligned} t_l(\theta,\phi) = \frac{\mu n_p(\theta,\phi)}{8\pi(1-\nu)}\sum_N \Big\{ \Delta u_i{}^{\beta} \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\rho}{r}\right)^2 \sin\vartheta [4\nu\delta_{lp}Q_{ik}^{\beta} r_{,k} \\ + (1-2\nu)(Q_{ip}^{\beta} r_{,l} + Q_{il}^{\beta} r_{,p} + \delta_{li}Q_{pk}^{\beta} r_{,k} + \delta_{pi}Q_{lk}^{\beta} r_{,k}) \\ + 3r_{,i}r_{,k}(Q_{pk}^{\beta} r_{,l} + Q_{lk}^{\beta} r_{,p})]\, d\xi_1\, d\xi_2 \Big\}_N \end{aligned} \tag{27}$$

where $\{\ \}_N$ designates that the expression in brackets is taken for the Nth surface element,

$$Q_{jk}^{\beta} \equiv n_j(\vartheta,\varphi)B_k{}^{\beta}(\vartheta,\varphi) - n_k(\vartheta,\varphi)B_j{}^{\beta}(\vartheta,\varphi) \tag{28}$$

$r_{,i}$, r/ρ, $\mathbf{n}$, ϑ, φ are given by the equations (15) and (17) and any repeated latin suffix is understood to be summed over the values 1, 2, 3, while greek one over, 1, 2, 3, 4.

4. CRACK CURVED IN THE SHAPE OF A PART OF CYLINDRICAL SURFACE

In this section the boundary integral equations (13) are rewritten approximately into a system of algebraic equations for the unknown crack opening at any nodal point on the surface, S_{cr}^{+}, of the crack shown in Fig. 2. Cylindrical coordinates (ρ, φ, z) are appropriate to use for the integration over S_{cr}^{+}. Then, Descartes coordinates of any point $\eta \in S_{cr}^{+}$ are given by

$$\begin{aligned} \eta_1 &= \rho\cos\varphi \\ \eta_2 &= \rho\sin\varphi \\ \eta_3 &= z \end{aligned} \tag{29}$$

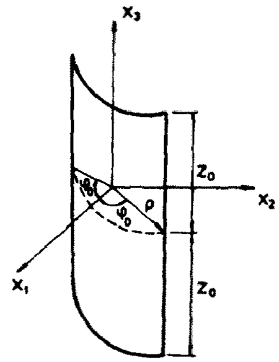

Fig. 2. The crack curved in the shape of a cylindrical surface.

where

$$\eta_1{}^2 + \eta_2{}^2 = \rho^2 = \text{const},\ \varphi \in \langle -\varphi_0; \varphi_0 \rangle,\ z \in \langle -z_0, z_0 \rangle.$$

The Descartes components of the gradient of the crack opening $\Delta u_{i,k}(\eta)$, defined on the crack surface, can be expressed through $\partial \Delta u_i / \partial \varphi$ and $\partial \Delta u_i / \partial z$ by the transformation matrix $A(\varphi, z)$

$$A = \begin{pmatrix} A_{1\rho} & -\dfrac{\sin\varphi}{\rho} & 0 \\ A_{2\rho} & \dfrac{\cos\varphi}{\rho} & 0 \\ A_{3\rho} & 0 & 1 \end{pmatrix} \tag{30}$$

as

$$\begin{pmatrix} \Delta u_{i,1} \\ \Delta u_{i,2} \\ \Delta u_{i,3} \end{pmatrix} = A \begin{pmatrix} 0 \\ \dfrac{\partial \Delta u_i}{\partial \varphi} \\ \dfrac{\partial \Delta u_i}{\partial z} \end{pmatrix}. \tag{31}$$

Let us denote the cylindrical coordinates of the point $\zeta \in S$ as (ρ, ϕ, Z). Then we have

$$\begin{gathered} r = |\zeta - \eta| = \rho[(\cos\phi - \cos\varphi)^2 + (\sin\phi - \sin\varphi)^2 + (Z - z)^2/\rho^2]^{1/2} \\ r_{,i} = \frac{\zeta_i - \eta_i}{r} = \frac{\rho}{r}\left(\cos\phi - \cos\varphi, \sin\phi - \sin\varphi, \frac{Z - z}{\rho}\right) \\ \mathbf{n}(\varphi) = (\cos\varphi, \sin\varphi, 0) \\ \mathrm{d}S_\eta = \rho\,\mathrm{d}\varphi\,\mathrm{d}z \end{gathered} \tag{32}$$

Inserting the expressions (31) and (32) in eqn (13), we obtain the integral equations in which all quantities depend on two variables φ, z. Since the BEM is to be used to solve these boundary integral equations, the change of variables

$$\begin{gathered} \varphi = \sum_{\alpha=1}^{4} N_\alpha(\xi_1, \xi_2)\varphi_\alpha \\ z = \sum_{\alpha=1}^{4} N_\alpha(\xi_1, \xi_2)\, z_\alpha \\ \xi_1, \xi_2 \in \langle -1; 1 \rangle \end{gathered} \tag{33}$$

is useful. As in the previous section, φ_α and z_α are coordinates of the αth nodal point of the surface element, φ and z are coordinates of any inner point of this element, and $N_\alpha(\xi_1, \xi_2)$ are given by the equation (18). Now the matrix

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} \dfrac{\partial z}{\partial \xi_2} & -\dfrac{\partial z}{\partial \xi_1} \\ -\dfrac{\partial \varphi}{\partial \xi_2} & \dfrac{\partial \varphi}{\partial \xi_1} \end{pmatrix} \tag{34}$$

yields the transformation

$$\begin{pmatrix} \dfrac{\partial \Delta u_i}{\partial \varphi} \\ \dfrac{\partial \Delta u_i}{\partial z} \end{pmatrix} = J^{-1} \begin{pmatrix} \dfrac{\partial \Delta u_i}{\partial \xi_1} \\ \dfrac{\partial \Delta u_i}{\partial \xi_2} \end{pmatrix} \tag{35}$$

From eqns (34) and (33)

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} z_\alpha \dfrac{\partial N_\alpha}{\partial \xi_2} & -z_\alpha \dfrac{\partial N_\alpha}{\partial \xi_1} \\ -\varphi_\alpha \dfrac{\partial N_\alpha}{\partial \xi_2} & \varphi_\alpha \dfrac{\partial N_\alpha}{\partial \xi_1} \end{pmatrix} \tag{36}$$

where the summation convection is understood.

The unknown crack opening displacement $\Delta u_i(\xi_1, \xi_2)$ are approximated according to eqn (22). The equations (36) and (22) convert the transformation relation (35) into

$$\begin{pmatrix} 0 \\ \dfrac{\partial \Delta u_i}{\partial \varphi} \\ \dfrac{\partial \Delta u_i}{\partial z} \end{pmatrix} = \frac{D_{\alpha\beta} \Delta u_i^\beta}{|J|} \begin{pmatrix} 0 \\ z_\alpha \\ -\varphi_\alpha \end{pmatrix} \tag{37}$$

where

$$D_{\alpha\beta} = \frac{\partial N_\alpha}{\partial \xi_2} \frac{\partial N_\beta}{\partial \xi_1} - \frac{\partial N_\alpha}{\partial \xi_1} \frac{\partial N_\beta}{\partial \xi_2}.$$

Finally, from eqns (30), (31) and (37)

$$\Delta u_{i,k} = \frac{1}{|J|} B_k^\beta \Delta u_i^\beta \tag{38}$$

where

$$\begin{pmatrix} B_1^\beta \\ B_2^\beta \\ B_3^\beta \end{pmatrix} = \frac{D_{\alpha\beta}}{\rho} \begin{pmatrix} -z_\alpha \sin\varphi \\ z_\alpha \cos\varphi \\ -\rho\varphi_\alpha \end{pmatrix} \tag{39}$$

Eventually, the equation (13) yields the system of algebraic equations for all Δu_i^β

$$\begin{aligned} t_l(\phi, Z) = \frac{\mu n_p(\phi)}{8\pi(1-\nu)\rho} \sum_N \Big\{ \Delta u_i^\beta \int_{-1}^{1} \int_{-1}^{1} \left(\frac{\rho}{r}\right)^2 [4\nu\delta_{lp} Q_{ik}^\beta r_{,k} \\ + (1-2\nu)(Q_{ip}^\beta r_{,l} + Q_{il}^\beta r_{,p} + \delta_{li} Q_{pk}^\beta r_{,k} + \delta_{pi} Q_{lk}^\beta r_{,k}) \\ + 3 r_{,i} r_{,k} (Q_{pk}^\beta r_{,l} + Q_{lk}^\beta r_{,p})] \, d\xi_1 \, d\xi_2 \Big\}_N \end{aligned} \tag{40}$$

where

$$Q^{\beta}_{jk} \equiv n_j(\varphi)B_k^{\beta}(\varphi, z) - n_k(\varphi)B_j^{\beta}(\varphi, z)$$

$r_{,i}$, $\partial r/\partial \rho$, $\mathbf{n}$, φ, z are given by (32) and (33), and any repeated latin suffix is understood to be summed over the values 1, 2, 3, while greek one over 1, 2, 3, 4.

5. SPHERICAL CRACK OF CIRCULAR CONTOUR UNDER AN AXIALLY-SYMMETRIC LOADING

Now, the system of algebraic equations, which can be used for numerical calculation of crack opening at all the nodal points of the spherical crack under an axial symmetric loading,

$$t_i(\theta, \phi)\Big|_{S^+_{cr}} = -t_i(\theta, \phi)|_{S^-_{cr}} = \delta_{i3}t_3(\theta) \tag{42}$$

is derived.

According to the geometrical and physical symmetry the problem is axially symmetric. Considering this symmetry, the problem is reduced to quasi-1-dimensional problem. Though this type of loading is included in more general problem, Sect. 3 deals with, we shall solve this problem considering its symmetry.

According to axial symmetry, we can write

$$\begin{aligned} \Delta u_1(\vartheta, \varphi) &= \Delta u(\vartheta)\cos\varphi \\ \Delta u_2(\vartheta, \varphi) &= \Delta u(\vartheta)\sin\varphi \\ \Delta u_3(\vartheta, \varphi) &= \Delta u_3(\vartheta) \end{aligned} \tag{43}$$

where $\Delta u(\vartheta) = \Delta u_1(\vartheta, 0)$ and $\Delta u(0) = 0$.

The unknown crack opening displacements $\Delta u(\vartheta)$' and $\Delta u_3(\vartheta)$ are to be calculated numerically. Consequently, n nodal points $\vartheta_\alpha(\alpha = 1, 2, \dots n)$ are chosen in the interval $\vartheta \in \langle 0: \ \vartheta_0\rangle$. Thus, one obtains $n-1$ elements on the arc $\rho = \text{const}$, $\varphi = \text{const} \equiv 0$. The αth element lies between the αth and $\alpha + 1$st nodal points. The crack opening displacements Δu^α and $\Delta u_3{}^\alpha$ are linearly approximated, within the αth element, by

$$\Delta u^\alpha(\vartheta) = (\Delta u^{\alpha+1} - \Delta u^\alpha)\xi + \Delta u^\alpha \tag{44a}$$

$$\Delta u_3{}^\alpha(\vartheta) = (\Delta_3^{\alpha+1} - \Delta u_3{}^\alpha)\xi + \Delta u_3{}^\alpha \tag{44b}$$

where

$$\xi = \frac{\vartheta - \vartheta_\alpha}{\vartheta_{\alpha+1} - \vartheta_\alpha}, \ \xi \in \langle 0; 1\rangle.$$

The gradients of the crack opening can be written by eqns (16), (43) and (44) as

$$(\Delta u_{i,k})^\alpha = \frac{\Delta u_i^{\alpha+1} - \Delta u_i^\alpha}{\vartheta_{\alpha+1} - \vartheta_\alpha}\frac{C^\alpha_{ik}}{\rho\sin\vartheta} \tag{45}$$

where both the suffixes i and α are not understood to be summed, and vectors C^α_{ik} are given by

$$\begin{pmatrix} c^\alpha_{11} \\ c^\alpha_{12} \\ c^\alpha_{13} \end{pmatrix} = \begin{pmatrix} \cos\vartheta\cos^2\varphi\sin\vartheta + \xi(\vartheta_{\alpha+1} - \vartheta_\alpha)\sin\varphi \\ \cos\vartheta\sin\vartheta\cos\varphi\sin\varphi - \xi(\vartheta_{\alpha+1} - \vartheta_\alpha)\sin\varphi\cos\varphi \\ -\sin^2\vartheta\cos\varphi \end{pmatrix}$$

$$\begin{pmatrix} c^\alpha_{21} \\ c^\alpha_{22} \\ c^\alpha_{23} \end{pmatrix} = \begin{pmatrix} \cos\vartheta\sin\vartheta\sin\varphi - \xi(\vartheta_{\alpha+1} - \vartheta_\alpha)\sin\varphi\cos\varphi \\ \cos\vartheta\sin\vartheta\sin^2\varphi + \xi(\vartheta_{\alpha+1} - \vartheta_\alpha)\cos^2\varphi \\ -\sin^2\vartheta\sin\varphi \end{pmatrix} \tag{46}$$

$$\begin{pmatrix} c_{31}^{\alpha} \\ c_{32}^{\alpha} \\ c_{33}^{\alpha} \end{pmatrix} = \begin{pmatrix} \cos\vartheta \sin\vartheta \cos\varphi \\ \cos\vartheta \sin\vartheta \sin\varphi \\ -\sin^2\varphi \end{pmatrix}.$$

and

$$\Delta u_1{}^{\alpha} = \Delta u_2{}^{\alpha} = \Delta u^{\alpha}$$
$$\vartheta = \vartheta_{\alpha} + \xi(\vartheta_{\alpha+1} - \vartheta_{\alpha}). \tag{47}$$

Combining the expression (45) with the equation (13), one obtains the system of algebraic equations for Δu^{α}, $\Delta u_3{}^{\alpha}$

$$(\alpha = 1, 2, \ldots, n)$$

$$\frac{8\pi\rho(1-\nu)}{\mu} t_l(\theta) = \sum_{\alpha=1}^{n-1} (\Delta u^{\alpha+1} - \Delta u^{\alpha}) I_l^{\alpha}(\theta) + \sum_{\alpha=1}^{n-1} (\Delta u_3^{\alpha+1} - \Delta u_3{}^{\alpha}) J_l^{\alpha}(\theta) \tag{48}$$

where the integrals $I_l^{\alpha}(\theta)$, $J_l^{\alpha}(\theta)$ are given by

$$I_l^{\alpha}(\theta) = \sum_{p,k=1}^{3} \sum_{i=1}^{2} n_p(\theta, 0) \int_0^1 \int_0^{2\pi} \left(\frac{\rho}{r}\right)^2 [4\nu\delta_{lp} Q_{ik}^{i\alpha} r_{,k}$$
$$+ (1-2\nu)(Q_{ip}^{i\alpha} r_{,l} + Q_{il}^{i\alpha} r_{,p} + \delta_{il} Q_{pk}^{i\alpha} r_{,k} + \delta_{ip} Q_{lk}^{i\alpha} r_{,k})$$
$$+ 3 r_{,i} r_{,k} (Q_{pk}^{i\alpha} r_{,l} + Q_{lk}^{i\alpha} r_{,p})]\, d\xi\, d\varphi \tag{49}$$

$$J_l^{\alpha}(\theta) = \sum_{p,k=1}^{3} n_p(\theta, 0) \int_0^1 \int_0^{2\pi} \left(\frac{\rho}{r}\right)^2 [4\nu\delta_{lp} Q_{3k}^{3\alpha} r_{,k}$$
$$+ (1-2\nu)(Q_{3p}^{3\alpha} r_{,l} + Q_{3l}^{3\alpha} r_{,p} + \delta_{l3} Q_{pk}^{3\alpha} r_{,k} + \delta_{p3} Q_{lk}^{3\alpha} r_{,k})$$
$$+ 3 r_{,i} r_{,k} (Q_{pk}^{3\alpha} r_{,l} + Q_{lk}^{3\alpha} r_{,p})]\, d\xi\, d\varphi \tag{50}$$

where the notations (46) and (47) are used, and further

$$Q_{jk}^{i\alpha} = n_j(\vartheta, \varphi) C_{ik}^{\alpha} - n_k(\vartheta, \varphi) C_{ij}^{\alpha}$$
$$\mathbf{n}(\nu, \varphi) = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta) \tag{51}$$
$$\frac{r}{\rho} = [(\sin\theta - \sin\vartheta\cos\varphi)^2 + \sin^2\vartheta \sin^2\varphi + (\cos\theta - \cos\vartheta)^2]^{1/2}$$
$$r_{,i} = \frac{\rho}{r}(\sin\theta - \sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\theta - \cos\vartheta)$$
$$\theta \in \langle 0; \vartheta_0 \rangle.$$

Since $t_1(\theta) = t_2(\theta) = 0$, the condition $\Delta u^1 = \Delta u(0) = 0$ can be used, and eqn (48) take the form

$$\frac{8\pi\rho(1-\nu)}{\mu} t_l(\theta) = -\sum_{\alpha=2}^{n-1} \Delta u^{\alpha}[I_l^{\alpha}(\theta) - I_l^{\alpha-1}(\theta)] + \Delta u^n I_l^{n-1}(\theta)$$
$$-\Delta u_3{}^1 J_l^1(\theta) - \sum_{\alpha=2}^{n-1} \Delta u_3{}^{\alpha}[J_l^{\alpha}(\theta) - J_l^{\alpha-1}(\theta)] + \Delta u_3{}^n J_l^{n-1}(\theta). \tag{52}$$

Once the system of algebraic equations (52) has been solved for the unknown $\Delta u_i{}^{\alpha}$, eqn (10) can be employed to obtain the displacement field $\mathbf{u}(\mathbf{x})$ at any point of the infinite elastic body.

The next application of the obtained crack opening displacement is the calculation of stress-intensity factors. Kassir and Sih[8] have shown that stress-intensity factors of a three-dimensional crack with a smooth contour can be evaluated by the relations which are

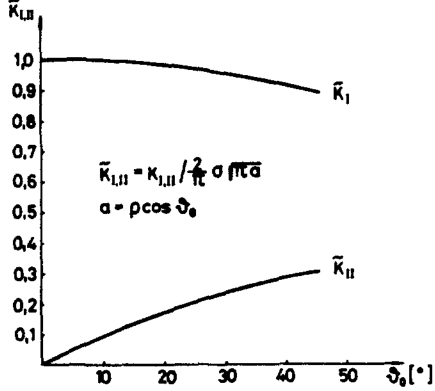

Fig. 3. Stress intensity factors $\tilde{K}_I$, $\tilde{K}_{II}$ for a spherical crack under constant loading.

valid for the case of plane strain. Then the stress-intensity factors for spherical crack (Fig. 1) are given by

$$K_{I,II} = \frac{E\sqrt{2\pi}}{8(1-\nu^2)} \frac{\Delta u_{I,II}}{\sqrt{\epsilon}} \tag{53}$$

where $\Delta u_{I,II}$ are the crack opening displacements in the directions I and II calculated at the point on the normal to the circular contour, and ϵ is the distance of this point from the contour.

The dependence of the stress-intensity factors $\mathbf{K}_{I,II}$, normalized to the stress intensity factor, K_{Icirc}, of penny-shaped crack, on the curvature is shown in Fig. 3 for the spherical crack of circular contour under a constant loading $t_i(\vartheta) = \sigma\delta_{i3}$ with $\sigma = \text{const}$. The dependence is the same as in the case of two-dimensional crack of the shape of circular arc [3].

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APPENDIX

The stress operator may be written in more compact form

$$\hat{T}_{ik}(n_x, \partial_x) = C_{ijkl} n_j(\mathbf{x})\, \partial_l \tag{A1}$$

where

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}),\ \lambda = \mu\frac{2\nu}{1-2\nu}. \tag{A2}$$

Using the apparent relation

$$\frac{\partial}{\partial x_k} U_{ij}(\mathbf{y}-\mathbf{x}) = -\frac{\partial}{\partial y_k} U_{ij}(\mathbf{y}-\mathbf{x})$$

and eqn (A1), eqn (12) may be rewritten as

$$t_l(\mathbf{x}) = c_{lpjr} c_{iskt} n_p(\mathbf{x}) \int_2 u_i(\boldsymbol{\eta}) n_s(\boldsymbol{\eta}) \partial_r' \partial_t' U_{kj}(\boldsymbol{\eta} - \mathbf{x}) \, \mathrm{d}S_\eta \tag{A3}$$

where

$$\partial_i' \equiv \frac{\partial}{\partial \eta_i}.$$

The integral (A3) may be decomposed[9] into a sum of surface and line integrals

$$t_l(\mathbf{x}) = t_l^1(\mathbf{x}) + t_l^2(\mathbf{x})$$

where

$$t_l^1(\mathbf{x}) = c_{lpjr} c_{iskt} n_p(\mathbf{x}) \int_s \kappa_{rs}^i \partial_t' U_{kj}(\boldsymbol{\eta} - \mathbf{x}) \, \mathrm{d}S_\eta \tag{A4}$$

$$\kappa_{rs}^i \equiv [n_r(\boldsymbol{\eta}) \partial_s' - n_s(\boldsymbol{\eta}) \partial_r'] u_i(\boldsymbol{\eta})$$

and the line integral $t_l^2(\mathbf{x})$ equals zero, if the surface S is closed. By differentiation of eqn (3) one obtains

$$\partial_t' U_{kj}(\boldsymbol{\eta} - \mathbf{x}) = \frac{1}{16\pi\mu(1-\nu) r^2} [(3 - 4\nu)\delta_{kj} r_{,t} - \delta_{jt} r_{,k} - \delta_{kt} r_{,j} + 3 r_{,k} r_{,j} r_{,t}]. \tag{A5}$$

After inserting (A2) and (A5) into (A4) and performing all summations, eqn (A4) takes the form

$$\begin{aligned} t_l(\mathbf{x}) = t_l^1(\mathbf{x}) = \frac{\mu u_p(\mathbf{x})}{8\pi(1-\nu)} \int_S \frac{1}{r_2} \{ & 4\nu \delta_{lp} \kappa_{ik}^i r_{,k} \\ & + (1 - 2\nu)[\kappa_{ip}^i r_{,l} + \kappa_{il}^i r_{,p} + r_{,k}(\kappa_{pk}^l + \kappa_{lk}^p)] \\ & + 3 r_{,i} r_{,k} (\kappa_{pk}^i r_{,l} + \kappa_{lk}^i r_{,p}) \} \, \mathrm{d}S_\eta. \end{aligned} \tag{A6}$$

As to the limit behaviour of $t_l^1(\mathbf{x})$, when $\mathbf{x} \to \boldsymbol{\zeta} \in S$, one can use theorems on limit behaviour of derivatives of simple layer potentials[10]. The continuity of $t_l^1(\mathbf{x})$, when $\mathbf{x} \to \xi \in S$, can be shown, provided that $u_i(\boldsymbol{\eta}) \in C^{1,\alpha}(S)$, $0 < \alpha \leq 1$. Then the integrals in $t_l^1(\zeta)$ are taken in the sense of the Cauchy principle value.